An MPEC Formulation for Parameter Identification of Complementarity Systems

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Using noisy observations of a dynamical multi-rigid-body system, determine the parameters of a given mixed complementarity problem dynamics model that best predicts the observation.
Problem Statement

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Using noisy observations of a dynamical multi-rigid-body system, determine the parameters of a given mixed complementarity problem dynamics model that best predicts the observation.

Coefficient of friction = 0.2
Let \( u \in \mathbb{R}^{n_1}, \ v \in \mathbb{R}^{n_2} \) and let \( g : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_1} \), \( f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}^{n_2} \) be two vector functions and the notation \( 0 \leq x \perp y \geq 0 \) imply that \( x \) is orthogonal to \( y \) and each component of each vector is non-negative.

**Definition**

The mixed complementarity problem is to find \( u \) and \( v \) satisfying

\[
g(u, v) = 0, \quad u \text{ free} \\
0 \leq v \perp f(u, v) \geq 0
\]

**Orthogonality**

\[
x = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix} + \begin{bmatrix} + \\ + \\ \vdots \end{bmatrix} = y
\]
Instantaneous Dynamics Model

\( \mathbf{M} \): Inertia matrix
\( \mathbf{W}_n \): Maps normal forces to body frame
\( \mathbf{W}_f \): Maps friction forces to body frame
\( \mathbf{q} \): Configuration
\( \mathbf{v} \): Velocity

\( \mathbf{\lambda}_{\text{vp}} \): Velocity product forces
\( \mathbf{\lambda}_{\text{app}} \): Applied forces
\( \mathbf{\lambda}_n \): Magnitude of normal forces
\( \mathbf{\lambda}_n \): Magnitude of frictional forces

- Newton-Euler Equations:
  \[ \mathbf{M}(\mathbf{q})\ddot{\mathbf{v}} = \mathbf{W}_n \mathbf{\lambda}_n + \mathbf{W}_f \mathbf{\lambda}_f + \mathbf{\lambda}_{\text{vp}} + \mathbf{\lambda}_{\text{app}} \]
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Dynamics Model

Instantaneous Dynamics Model

\[ G = \text{Jacobian of kinematic velocity map} \]

- Newton-Euler Equations:
  \[ M(q)\dot{\nu} = W_n\lambda_n + W_f\lambda_f + \lambda_{vp} + \lambda_{app} \]

- Kinematic Map: \( \dot{q} = G\nu \)
\( \psi_{in} = \text{signed distance function at contact } i \)

- **Newton-Euler Equations:**
  \[
  M(q) \ddot{\nu} = W_n \lambda_n + W_f \lambda_f + \lambda_{vp} + \lambda_{app}
  \]

- **Kinematic Map:**
  \( \dot{q} = G \nu \)

- **Normal Complementarity Constraint:**
  \( 0 \leq \lambda_{in} \perp \psi_{in} (q, t) \geq 0 \)
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**Dynamics Model**

### Instantaneous Dynamics Model

- **$v_i$**: relative velocity at contact $i$
- **$F_i(\lambda_{in}, \mu)$**: friction cone at contact $i$

- **Newton-Euler Equations**:  
  \[ M(q)\dot{\nu} = W_n\lambda_n + W_f\lambda_f + \lambda_{vp} + \lambda_{app} \]

- **Kinematic Map**:  
  \[ \dot{q} = G\nu \]

- **Normal Complementarity Constraint**:  
  \[ 0 \leq \lambda_{in} \perp \psi_{in}(q, t) \geq 0 \]

- **Max. Power Dissipation**:  
  \[ \lambda_{if} \in \arg\max\{-v_{if}\lambda'_{if} : \lambda'_{if} \in F_i(\lambda_{in}, \mu)\} \]
Discrete Time Dynamics Model

\[ \dot{\nu} \approx (\nu^{\ell+1} - \nu^\ell)/h \qquad \dot{q} \approx (q^{\ell+1} - q^\ell)/h \]

\[ M \nu^{\ell+1} = M \nu^\ell + h(W_n \lambda_n^{\ell+1} + W_f \lambda_f^{\ell+1} + \lambda_{vp}^\ell + \lambda_{app}^\ell) \]

\[ q^{\ell+1} = q^\ell + hG \nu^{\ell+1} \]

\[ 0 \leq h \lambda_n^{\ell+1} \perp \psi_n(q^{\ell+1}) \geq 0 \]

\[ \lambda_{if}^{\ell+1} \in \arg\max \{ -v_1^{\ell+1} \lambda_{if}' : \lambda_{if}^{\ell+1} \in F_i(\lambda_{in}^{\ell+1}, \mu) \} \]

Where \( h \) is the length of the time step and superscripts \( \ell \) and \( \ell + 1 \) denote values at the beginning and end of the \( \ell \)th time step.
The dynamic system is modeled with two equations:

State Transition Equation

\[ x^{\ell+1} = F(x^{\ell}, u^{\ell}, \zeta^{\ell}) \]

\( F(\cdot) \) : Dynamic model

\( x^{\ell} \) : Unobserved state at time \( \ell \)

\( u^{\ell} \) : known input at time \( \ell \)

\( \zeta^{\ell} \) : process noise at time \( \ell \)
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Estimation Problem

State Estimation

Measurement Equation

\[ y^\ell = H(x^\ell, n^\ell) \]

\( H(\cdot) \) : Measurement Function

\( x^\ell \) : Unobserved state at time \( \ell \)

\( y^\ell \) : Observed state at time \( \ell \)

\( n^\ell \) : Observation noise at time \( \ell \)

\[ y = [x_p \ y_p] \]

\[ \Leftrightarrow H(x^\ell, n^\ell) = [x^\ell_1 \ x^\ell_2] + n^\ell \]

\[ x^\ell = MCP(x^0) \]
Parameter Estimation

Determine the nonlinear mapping:

\[ y^\ell = G(x^\ell, p) \]

\( G(\cdot) \): Nonlinear Map

\( p \): Parameters of the mapping

Coulomb's law: \( \lambda_f \leq \mu \lambda_n \)

\[ 2\tan^{-1}\mu \]

\[ W_n \lambda_n \]
Dual Estimation

Special case where both the state and parameters must be learned simultaneously.

\[ x^{\ell+1} = F(x^\ell, u^\ell, \zeta^\ell, p) \]
\[ y^\ell = H(x^\ell, u^\ell, n^\ell, p) \]

Both \( x^\ell, \ell = 1, 2, \ldots, N \) and \( p \) must be simultaneously estimated.

\[ x^\ell = [x_p^\ell \ y_p^\ell \ \dot{x}_p^\ell \ \dot{y}_p^\ell \ \lambda_n^\ell \ \lambda_f^\ell \ \sigma^\ell] \]
\[ y^\ell = [x_p^\ell \ y_p^\ell] \]
\[ p = [\mu] \]

\( F(\cdot) = \text{Discrete time MCP} \)
\[ H(\cdot) = [x_p^\ell \ y_p^\ell] + n^\ell \]
Difficulties of Current Approaches

Difficulties of Kalman Filtering
- Not possible to apply physical constraints to parameters or state (e.g. $\mu > 0$)
- Noise is assumed to be Gaussian
- Fails with multimodal pdfs

Difficulties of Particle Filtering
- Difficult to apply physical constraints to parameters or state
- With small process noise, all particles can collapse into a single point within a few iterations
### Problem Formulation

**Optimization Problem for Dual Estimation of Rigid Body Dynamics**

\[
\begin{align*}
\min_{n^0, \ldots, n^N, x^0, \ldots, x^N, p} & \quad (x^0 - \bar{x}^0)^T (x^0 - \bar{x}^0) + \sum_{\ell=0}^{T} n^\ell T n^\ell \\
\text{subject to:} & \quad p \in \mathcal{P}, \quad n \in \mathcal{N} \quad (2) \\
& \quad x^{\ell+1} \in SOL(MCP(x^\ell, p)) \quad (3) \\
& \quad y^\ell = [I \ 0] x^\ell + n^\ell \quad (4)
\end{align*}
\]

where \( \bar{x}^0 \) is the initial state estimate, \( n \) is a slack variable representing the error between observation and prediction, \( I \) is an identity matrix of appropriate size, MCP is the mixed complementarity problem arising from the discrete time dynamics model, and \( \mathcal{P} \) and \( \mathcal{N} \) are the sets of possible parameter values and max observations error respectively.
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MPEC Definition

**Definition**

\[
\min_{u \in \mathbb{R}^{n_1}, v \in \mathbb{R}^{n_2}} f(u, v)
\]

subject to: \((u, v) \in Z\), and

\[
v \text{ solves } \text{MCP}(g(u, \cdot), B),
\]

where \(f\) is a desired objective function, \(Z \subseteq \mathbb{R}^{n_1 + n_2}\) is a nonempty closed set (equation (6) represents standard nonlinear programming constraints), and equation (7) signifies \(v\) is a solution to the MCP defined by the function \(g\) and the bound set \(B\).

For the special case when \(f\) and the MCP are linear, the problem is known as a linear program with equilibrium constraints (LPEC).
If we wish to remove the nonlinearties, we would be forced to make the following assumptions:

- Rotations are restricted to being “small” because the contact Jacobians are a function of the bodies' states.
- We must assume that either the parameters or state values are known to eliminate the bilinear constraints. For example, the friction model contains the bilinear constraint: $\mu \lambda_{n}^{\ell+1}$
We use the AMPL mathematical modeling language to formulate the MPECs, and the nonlinearly constrained optimization solvers available on the NEOS Server.
Goal: Estimate the coefficient of friction ($\mu$) from the “noisy” observations. Noise was added by uniformly sampling $[-\epsilon, \epsilon]$. 
Estimation Formulation

State: \( \mathbf{x} = \begin{bmatrix} x_p & y_p & \dot{x}_p & \dot{y}_p & \lambda_n & \lambda_f & \sigma \end{bmatrix} \)

Observations: \( \mathbf{y} = \begin{bmatrix} x_p & y_p & \dot{x}_p & \dot{y}_p \end{bmatrix} \)

Parameters: \( \mathbf{p} = [\mu] \)

We cannot observe the velocity directly, but can approximate it.

\[ \mathbf{x}^{\ell+1} \in SOL(MCP(\mathbf{x}^{\ell})) \quad (Described \ on \ next \ slide) \quad (8) \]

\[ \mathbf{y}^{\ell} = \begin{bmatrix} I_{4 \times 4} & 0_{4 \times 4} \end{bmatrix} \mathbf{x}^{\ell} + \mathbf{n}^{\ell} \quad (9) \]

\[ \mathcal{P} = 0 \leq \mu \leq 1 \quad (10) \]

\[ \mathcal{N} = \mathbf{n}^{\ell \top} \mathbf{n}^{\ell} \leq \epsilon^2 \quad (11) \]
Equations of Motion

\[ 0 = q^{\ell+1} - q^\ell - h\nu^{\ell+1} \]
\[ 0 = M(\nu^{\ell+1} - \nu^\ell) - h(W_n\lambda_n^{\ell+1} + W_f\lambda_f^{\ell+1} + \lambda_{\text{app}}) \]
\[ 0 \leq \lambda_n^{\ell+1} \perp y^{\ell+1} \geq 0 \quad \text{Normal contact model} \]
\[ 0 \leq \lambda_f^{\ell+1} \perp E\sigma^{\ell+1} + W_f^T\nu^{\ell+1} \geq 0 \quad \text{Friction Model} \]
\[ 0 \leq \sigma^{\ell+1} \perp \mu\lambda_n^{\ell+1} - E^T\lambda_f^{\ell+1} \geq 0 \]

where \( q = \begin{bmatrix} x_p \\ y_p \end{bmatrix} \), \( \nu = \begin{bmatrix} \dot{x}_p \\ \dot{y}_p \end{bmatrix} \), \( M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \), \( W_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \),
\( W_f = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \), \( E = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), and \( \lambda_{\text{app}} = \begin{bmatrix} x_{\text{app}} \\ -mg \end{bmatrix} \)

4 equations and 4 complementarity constraints per time step.
MPEC Formulation

\[
\min_{x^0, \ldots, x^N} \ (x^0 - \bar{x}^0)^T (x^0 - \bar{x}^0) + \sum_{\ell=0}^{N} n^\ell^T n^\ell
\]

subject to:

\[
\begin{aligned}
0 &= q^{\ell+1} - q^\ell - h\nu^{\ell+1} \\
0 &= M(\nu^{\ell+1} - \nu^\ell) - h(W_n\lambda_n^{\ell+1} + W_f\lambda_f^{\ell+1} + \lambda_{app}) \\
0 &\leq \lambda_n^{\ell+1} \perp y^{\ell+1} \geq 0 \\
0 &\leq \lambda_f^{\ell+1} \perp E\sigma^{\ell+1} + W_f^T \nu^{\ell+1} \geq 0 \\
0 &\leq \sigma^{\ell+1} \perp \mu\lambda_n^{\ell+1} - E^T \lambda_f^{\ell+1} \geq 0 \\
0 &\leq \mu \leq 1 \\
n^\ell^T n^\ell &\leq \epsilon^2 \\
y^\ell &= [I_{4\times4} \ 0_{4\times4}] x^\ell + n^\ell
\end{aligned}
\]

\[
\text{4N equations, 4N complementarity, 4N + 1 inequalities, 4N equations}
\]
Examples

2D Particle Falling and Sliding

Results

Values used in simulation: \( \mu = 0.2, \, h = 0.05, \, q^0 = [0, 3], \lambda_{app} = [5, -9.81m], \, m = 1, \, N = 100 \)

<table>
<thead>
<tr>
<th>Observe. Error</th>
<th>( \mu )</th>
<th>Obj. Val</th>
<th>Iters</th>
</tr>
</thead>
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<td>5.00E-05</td>
<td>0.2</td>
<td>2.03e-07</td>
<td>301</td>
</tr>
<tr>
<td>5.00E-04</td>
<td>0.2</td>
<td>1.61e-05</td>
<td>443</td>
</tr>
<tr>
<td>5.00E-03</td>
<td>0.2</td>
<td>1.467e-03</td>
<td>19</td>
</tr>
<tr>
<td>5.00E-02</td>
<td>0.200022</td>
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<td>149</td>
</tr>
<tr>
<td>5.00E-01</td>
<td>0.199873</td>
<td>1.48e+01</td>
<td>96 (infeasible pt)</td>
</tr>
</tbody>
</table>

Table: Solver: KNITRO

Initial guess for MPEC: \( q^l = q^\ell, \, \lambda^n_l = 0, \, \lambda_f^l = 0, \, \sigma^l = 0, \, \mu = 1 \).
We extend the previous example to now include multiple particles starting with random initial positions \((x \in [-10, 10], y \in [0, 5])\) and random coefficients of friction \((\mu \in (0, 0.5])\).
Measurement noise $\in [-0.005, 0.005]$.

<table>
<thead>
<tr>
<th># Part.</th>
<th>Num. Var.</th>
<th>$\mu$ Error</th>
<th>Obj. Val</th>
<th>Iters</th>
</tr>
</thead>
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<td>2</td>
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<td>0</td>
<td>4.09e-03</td>
<td>334</td>
</tr>
<tr>
<td>3</td>
<td>3009</td>
<td>0</td>
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<td>185</td>
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<tr>
<td>5</td>
<td>5015</td>
<td>0.000006</td>
<td>1.05e-02</td>
<td>364</td>
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<tr>
<td>10</td>
<td>10030</td>
<td>0.0000072</td>
<td>1.66e-02</td>
<td>281</td>
</tr>
</tbody>
</table>

Table: Solver: KNITRO

$\mu$ error is the root mean squared error. Initial guess for MPEC: $q^l = \tilde{q}^l$, $\lambda_n^l = 0$, $\lambda_f^l = 0$, $\sigma^l = 0$, $\mu = 1$. 
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Examples

Sliding Block Scene

Experimental set up
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Examples

Experimental Sliding Block

Goal

Goal is to determine the surface friction coefficient assuming a fixed support tripod with linearized friction cones.

Surface friction support tripod
Comparison of trajectory between particle filter, MPEC and observation.
Comparison of velocity between particle filter, MPEC and observation
Results

Bootstrap filtering result for $\mu_s$

Mean of particle filter’s estimate of $\mu = 0.3246$

MPEC estimate of $\mu = 0.330311$

Surface friction estimate of the particle filter.
Future Work

- Online Solution
  - Partial information
  - Moving horizon framework
- Observability of nonlinear and nonsmooth system
ACKNOWLEDGMENT

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